EXERCISE 1

SAME MARKING SCHEME AS HOMEWORKS

1) MANY POSSIBLE ANSWERS, CLAIMS SHOULD BE JUSTIFIED. FOR EXAMPLE

An immerse submanifold is not necessarily a submanifold as a set: being an immersion is a property of the differential of a function after all, the function itself does not need to be injective (which means that its image can have self intersections).

An embedded submanifold is an immersed submanifold for which the inclusion is a topological embedding: an homeomorph. onto its image. This ensures that the subspace topology is the same as the submanifold topology and then that the embedded submanifold is a manifold.

REMARK: the same can be justified with carefully crafted examples or using pictures!

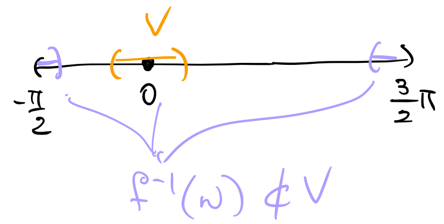
2) It is a matter of checking the definitions

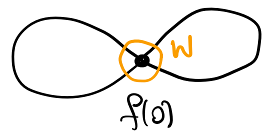
4 pts for correct use of definitions

+ 2 pts for each example w. reasonably argued immersion/embedding (or lack thereof)

$$Df(t) = (-\sin t, 2\cos(2t), 0)$$
$$Dg(t) = (1, 3t^2, 0)$$
$$Dh(t) = (\cos(t), -\sin(t), 1)$$

The map f, restricted to $\mathcal{U}$, is injective (is ok if they justify it with a figure) and the same for the Jacobian, so $f(\mathcal{U})$ is the image of injective immersion. Unfortunately, with subspace topology from $\mathbb{R}^2$ the image is not homeomorphic to $\mathbb{R}^2$. This can be seen from the fact that for any small nbd of 0, there is no nbd of 0 in the image whose preimage is in it





(this last part is a tricky point, only remove 1 point if incorrect)

The map g is injective and its differential is as well. Moreover it is an homeomorphism onto its image (inverse $(t, t^3, 0) \mapsto t$ is continuous)

$\Rightarrow$ embedded submanifold

The map h is injective, its differential is injective and h is a homeomorphism: $h^{-1}(x,y,z) = z$

$\Rightarrow$ embedded submanifold

In the last two one can also argue that g and h are the graphs of continuous functions and quote a thm from the notes.

EXERCISE 2

- $$\phi^* \omega = (z\circ\phi)(y\circ\phi)^2\, d(x\circ\phi)\wedge d(y\circ\phi)$$
$$= t\,(u+v)^2\, d(2uv)\wedge d(u+v)$$
$$= t\,(u+v)^2\, 2\,(v\,du + u\,dv)\wedge(du+dv)$$
$$= 2t(u+v)^2\,(v\,du\wedge dv + u\,dv\wedge du)$$
$$= 2t\,(u+v)^2\,(v-u)\,du\wedge dv$$

- $$\mathcal{L}_X\omega = \iota_X(d\omega) + d(\iota_X\omega)$$
$$= \iota_X(y^2\,dx\wedge dy\wedge dz) + d\Big(zy^2\,dx\big(\tfrac{\partial}{\partial y}\big)^{0}dy - zy^2\,dy\big(\tfrac{\partial}{\partial y}\big)dx\Big)$$

other term vanish due to $dy\wedge dy=0$

$$= \underbrace{-\,y^2\,dx\wedge dz}_{\substack{\text{either computed explicitly or justified}\\ \text{vie } \iota_X\,\alpha\wedge\beta = (\iota_X\alpha)\wedge\beta + (-1)^k\alpha\wedge(\iota_X\beta)\\ \text{and the fect thet only one term can be nonzero}}} - d(zy^2\,dx)$$

$$= \cancel{-y^2\,dx\wedge dz} + \cancel{y^2\,dx\wedge dz} + 2yz\,dx\wedge dy$$

$-dz\wedge dx$ $\qquad$ $-dy\wedge dx$

This is over justified but they need to show some steps to justify the computation.

Remove some points for obscure steps or for conceptual mistakes, overall 1-2 pts for computational mistakes

EXERCISE 3

- $X = \nabla f = \# df$

$$\Rightarrow \nabla \times X = \# \star d \flat \nabla f$$

$$= \# \star d \underbrace{\flat \#}_{id} df = \# \star \underbrace{d^2}_{0} f = 0$$

isomorphisms

- let $\nabla \times X = 0$ then $\# \star d \flat = 0$

$$\underbrace{\# \star}_{\text{both isomorphisms}} d X^\flat = 0$$

$$\Rightarrow dX^\flat = 0$$

Since we are in $\mathbb{R}^3$ closed and exact forms coincide $\Rightarrow X^\flat$ must be exact leading to the claim

10 points for complete answer

remove points for missing details/explanations

! A compact but justified solution is perfectly fine

EXERCISE 4

2 • First, X must be nonzero, otherwise $\iota_X \omega \neq 1$ and ω is dual basis element to X by definition

• $\underset{\overset{\shortparallel}{0}}{\mathcal{L}_X \omega} = i_X(d\omega) + d(\underset{\overset{\shortparallel}{1}}{\iota_X \omega}) = i_X(d\omega)$

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2 ⇒ Consider a frame $(X, e_1, \dots, e_{n-1})$ of TM with e_i independent from X

2 $\Rightarrow \iota_X(\omega \wedge d\omega) = d\omega - \omega \wedge (\underbrace{\iota_X d\omega}_{0})$

$= d\omega$ nowhere vanishing

2 since it is enough to show that there are some vector fields for which it is not zero: X with any two orthogonal vector fields will do the trick

Not strict, as long as the answer is correct & justified this is fine

• $\omega \wedge d\omega$ is a volume form ⇒ M orientable

10 pts if correctly justified

EXERCISE 5

Remove 1-2 pts for computational mistakes, more points for conceptual mistakes

- $\int_{\mathbb{S}^2} \omega = \int_{\mathbb{B}^2} d\omega = \int_{\mathbb{B}^2} 3\, dx^1 \wedge dx^2 \wedge dx^3 = 3 \cdot \underbrace{\frac{4}{3}\pi}$

$\mathbb{S}^2 = \partial \mathbb{B}^2$

ok if they write vol (Ball)

Full computation is much longer but ok of course

- If ω is exact $\Rightarrow \omega = d\eta$ and

$\cancel{4}\pi = \int_{\mathbb{S}^2} d\eta \neq \int_{\partial \mathbb{S}^2} \eta = \phi$

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About closedness: $d\omega$ is a 3-form on a 2-dim manifold $\mathbb{S}^2$ $\Rightarrow$ must be 0.

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